Weak Interaction Effects in Electrodynamics*

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A perturbation calculation with cutoff is performed for electron scattering in an external electromagnetic field to lowest order in the weak interactions, assuming an intermediate vector meson. Renormalization is based on recent general results of Sekine in a theory with a parity nonconserving interaction. The renormalized S matrix is then used to obtain a close estimate of the parity impurities present in atomic and molecular states due to the weak interactions of the electrons. In particular the 2^2P admixture to the 2^2S state of hydrogen is found to be about 10^{-12} for reasonable values of the parameters. The calculation agrees with general considerations on electromagnetic properties of leptons advanced by Zel'dovich and Perelomov, but yields an estimate of the hydrogen amplitude smaller by 10^{-3}. A theoretical upper limit of 10^{-18} on the parity impurity of the ground state of O_2 is suggested, well below the experimental upper limit of 3×10^{-8}.

1. INTRODUCTION

HISTORICALLY, the accepted weak interaction Hamiltonian has been built up more or less *a posteriori* by including a term for each observed decay process. Thus, β decay was well accounted for by a term $G\bar{\psi}_p\gamma_\alpha\frac{1}{2}(1+\gamma_5)\psi_n\bar{\psi}_e\gamma_\alpha\frac{1}{2}(1+\gamma_5)\psi_\nu$; while μ decay required $G\bar{\psi}_\mu\gamma_\alpha\frac{1}{2}(1+\gamma_5)\psi_\nu\bar{\psi}_\nu\gamma_\alpha\frac{1}{2}(1+\gamma_5)\psi_e$, and so on for the strange particles.

However, many have thought it more appealing to be able to assume an appropriate weak "current," and to postulate some kind of coupling to form the interaction Lagrangian density. At the very minimum such a current would have to include $ig\bar{\psi}_n\gamma_\alpha\frac{1}{2}(1+\gamma_5)\psi_p$, $ig\bar{\psi}_\mu\gamma_\alpha\frac{1}{2}(1+\gamma_5)\psi_\nu$ and $ig\bar{\psi}_e\gamma_\alpha\frac{1}{2}(1+\gamma_5)\psi_\nu$ in order to account for β decay and μ decay in one scheme.

The question then arises how in detail the current is coupled. The first possibility would be direct coupling with $J^*\cdot J$ for the interaction Lagranigan density, since it contains the usual Fermi interactions as above. A more interesting suggestion has been made by various authors that this current is coupled to a quite massive, charged vector-meson field. In favor of such an idea is the close analogy with electrodynamics, and a less striking analogy with meson theory of nuclear forces.

The meson hypothesis has been seriously in doubt until very recently due to the failure to observe the decay $\mu\rightarrow e+\gamma$.[1] The latest experimental upper limit on the branching ratio $\rho=R(\mu\rightarrow e+\gamma)/R(\mu\rightarrow e+\nu+\bar{\nu})$ is 6×10^{-8} at the 90% confidence level.[2] This is to be compared with a prediction of $\rho\approx10^{-4}$ on the basis of the intermediate-vector-meson hypothesis. But, recent evidence tends to confirm that there is an independent neutrino field associated with the muon, as well as one with the electron.[3] This would mean that the unobserved μ decay could not occur to lowest order in a vector meson theory of weak interactions.

Thus, at present there is no compelling evidence against a heavy, charged intermediate-vector meson. At the same time such a theory would remove some of the difficulties in the usual first-order Fermi theory, e.g., the high-energy violation of unitarity in scattering. To settle the question it would be most convincing to produce vector mesons in some sort of inverse process, such as $\bar{\nu}\rightarrow X^-+e^+$; and this may be possible before long in the high-energy neutrino experiments. But, another interesting possibility of observing the meson indirectly exists in the slight parity nonconservation (PNC), which weak interactions introduce into electromagnetic interactions of leptons.

Although the lowest order diagrams leading to $\mu\rightarrow e+\gamma$ are forbidden by the existence of two neutrinos, very similar diagrams are allowed in the electromagnetic scattering of electrons. Zel'dovich and Perelomov[4] have dealt with the order of magnitude of these corrections. They predict that in the meson theory the corrections are much larger than in a direct coupling theory, because two meson-lepton vertices are of the same order as a single four-fermion vertex. Now electromagnetic corrections occur only due to intermediate states. In any theory at least two vertices are needed to provide such a state, which makes the corrections in the direct coupling theory of order G^2—G being the weak interaction coupling constant, about $10^{-5}\ M_n^{-2}$. But in a meson theory the meson itself provides the intermediate state, and the necessary two vertices make the corrections of order G. This great enhancement of PNC effects in electrodynamics may provide a sensitive check on the existence of such a meson.

Thus, it seems desirable at this time using the intermediate-vector-meson hypothesis actually to calculate the magnitude of these PNC corrections to electron scattering. Then, using the scattering matrix it is possible to obtain a close estimate of parity impurities in atomic states.

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2. MESON THEORY OF WEAK INTERACTIONS (THE ELECTRON PART OF THE CURRENT)

Consider a theory containing electrons, photons, neutrinos, and charged vector mesons. The free Lagrangian density for the theory will be taken as[5]

$$\mathfrak{L}_0=-\tfrac{1}{2}[\bar{\psi}_\nu\gamma_\alpha\overleftrightarrow{\partial}_\alpha\psi_\nu+\bar{\psi}_e(\gamma_\alpha\overleftrightarrow{\partial}_\alpha+2m)\psi_e] -\tfrac{1}{2}(\partial_\beta\phi^*{}_\alpha-\partial_\alpha\phi^*{}_\beta)(\partial_\beta\phi_\alpha-\partial_\alpha\phi_\beta) -M^2\phi^*{}_\alpha\phi_\alpha-\tfrac{1}{4}F_{\alpha\beta}F_{\alpha\beta}; \quad (1)$$

$F_{\alpha\beta}$ here is the familiar electromagnetic field tensor.

The sum of the electromagnetic interaction Lagrangian of the electron, well established from electrodynamics, and that of a vector meson whose gyromagnetic ratio is $1+\mu$ and whose quadrupole moment is μ is

$$\mathfrak{L}_{\rm em}=\tfrac{1}{2}ie[\bar{\psi}_e,(\mathfrak{A}\cdot\gamma)\psi_e]-ie:[(\phi^*{}_\alpha\overleftrightarrow{\partial}_\beta\phi_\alpha)\mathfrak{A}_\beta+(\partial_\alpha\phi^*{}_\alpha) \times(\phi_\beta\mathfrak{A}_\beta)+(\phi^*{}_\alpha\mathfrak{A}_\alpha)(\partial_\beta\phi_\beta)+(1+\mu)\phi^*{}_\alpha F_{\alpha\beta}\phi_\beta]: \quad (2)$$

Now the electron contributes to the vector and axial vector currents a term $J_\alpha{}^e=ig\bar{\psi}_e\gamma_\alpha\frac{1}{2}(1+\gamma_5)\psi_\nu$. The coupling constant g is related to G for a particular coupling scheme from, for example, a comparison of the calculated μ-decay rates. If the current is coupled to a meson of mass M, this comparison yields $g^2/M^2=8^{1/2}G$. In analogy with the $j\cdot A$ interaction Lagrangian of electrodynamics, the Lagrangian for weak interactions will be taken as $(J_\alpha\phi_\alpha+J^*{}_\alpha\phi^*{}_\alpha)$. Including only the electron part of the current gives

$$\mathfrak{L}_w=ig[\bar{\psi}_e(\gamma_\alpha\phi_\alpha)\tfrac{1}{2}(1+\gamma_5)\psi_\nu+\bar{\psi}_\nu\tfrac{1}{2}(1-\gamma_5)(\gamma_\alpha\phi^*{}_\alpha)\psi_e]. \quad (3)$$

It is known that such a theory is not renormalizable in perturbation theory by ordinary methods, since the meson propagator in momentum space is of order 1 as the internal-meson momentum k becomes infinite, rather than of order k^{-2} as in electrodynamics. But the Fermi theory is not renormalizable either. Thus, based on general considerations, renormalization will be carried out in perturbation theory to the order of concern, using a cutoff.

The S operator is expanded in the usual manner for perturbation theory.

$$S=\sum_{n=0}^{\infty}(-i)^n(n!)^{-1}\int d^4x_1\cdots \times\int d^4x_n\mathbf{T}\{\mathfrak{L}_{\rm int}{}^{\rm (in)}(x_1)\cdots\mathfrak{L}_{\rm int}{}^{\rm (in)}(x_n)\}. \quad (4)$$

The use of the interaction Lagrangian in the S operator requires that the propagator for the vector meson be taken as $-i(\delta_{\alpha\beta}-M^{-2}\partial_\alpha\partial_\beta)\Delta_F(x)$ with $\Delta_F(x)$ the ordinary Feynman function for mass M.

[5] Hermitian γ matrices are used. The scalar product of two 4 vectors, indicated by $\mathbf{V}\cdot\mathbf{W}$, represents $\mathbf{V}\cdot\mathbf{W}-V_0W_0$ with V_0 and W_0 the real fourth components.

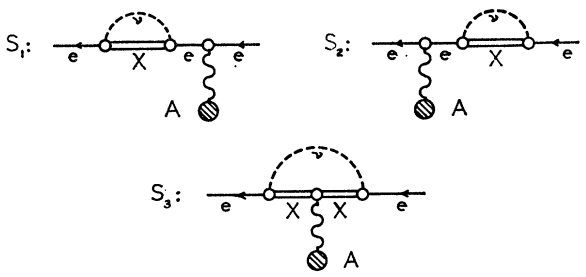

FIG. 1. Feynman diagrams to order g^2 in the PNC meson theory of weak interactions for the electron scattering matrix in an external electromagnetic field, represented by a wavy line terminating in a cross-hatched circle.

3. LOWEST ORDER MATRIX ELEMENTS

Beside the usual diagrams for electron scattering including radiative corrections, the three diagrams in Fig. 1 contribute to lowest order in g. (Diagrams of order g^2e^4 are neglected.)

If p and p' are the initial and final momenta of the electron respectively and ω is the momentum transfer to the external field, the S-matrix elements corresponding to these diagrams are

$$\begin{aligned}S_1&=-em(2\pi p_0'p_0)^{-1/2}\bar{U}(\mathbf{p}') \times\Sigma(p')(m+i\gamma\cdot p')^{-1}\gamma\cdot A(\omega)U(\mathbf{p}),\\ S_2&=-em(2\pi p_0'p_0)^{-1/2}\bar{U}(\mathbf{p}')\gamma \cdot A(\omega)(m+i\gamma\cdot p)^{-1}\Sigma(p)U(\mathbf{p}), \quad (5)\\ S_3&=-em(2\pi p_0'p_0)^{-1/2}\bar{U}(\mathbf{p}')A_\alpha(\omega)\Lambda_\alpha(p',p)U(\mathbf{p}),\end{aligned}$$

where

$$\Sigma(p)=g^2(2\pi)^{-4}\tfrac{1}{2}(1-\gamma_5) \int d^4k[-2\gamma\cdot k+M^{-2}\gamma\cdot(p-k)\gamma\cdot k\gamma\cdot(p-k)] \times(k^2-i\epsilon)^{-1}[M^2+(p-k)^2-i\epsilon]^{-1},$$

and

$$\begin{aligned}\Lambda_\alpha(p',p)=&-ig^2(2\pi)^{-4}\tfrac{1}{2}(1-\gamma_5) \times\int d^4kD(k)\{[\delta_{\sigma\tau}(p'+p-2k)_\alpha-(p'-k)_\tau\delta_{\sigma\alpha}\\ &-(p-k)_\sigma\delta_{\tau\alpha}+(1+\mu)(\omega_\sigma\delta_{\tau\alpha}-\omega_\tau\delta_{\sigma\alpha})]\\ &\times[\gamma_\tau+M^{-2}(p'-k)_\tau\gamma\cdot(p'-k)](\gamma\cdot k) \times[\gamma_\sigma+M^{-2}(p-k)_\sigma\gamma\cdot(p-k)]\},\end{aligned}$$

with

$$D(k)=[M^2+(p'-k)^2-i\epsilon]^{-1}(k^2-i\epsilon)^{-1} \times[M^2+(p-k)^2-i\epsilon]^{-1}.$$

The Lorentz gauge must be employed in order to use the familiar Feynman rules for the electron and photon parts of the diagrams; hence, $\partial_\alpha A_\alpha(x)=0$, or $\omega_\alpha A_\alpha(\omega)=0$ in momentum space. Applying this condition and the

identity $p\cdot p'=-m^2-\frac{1}{2}\omega^2$, one obtains for $A_\alpha\Lambda_\alpha$ the expression

$$A_\alpha(\omega)\Lambda_\alpha(p',p)=-ig^2(2\pi)^{-4}\tfrac{1}{2}(1-\gamma_5)\int d^4kD(k)\{A\cdot(p+p'-2k)[-2\gamma\cdot k+M^{-2}\gamma\cdot(p'-k)(\gamma\cdot k)\gamma\cdot(p'-k)$$
$$+M^{-2}\gamma\cdot(p-k)(\gamma\cdot k)\gamma\cdot(p-k)-M^{-2}(1+\tfrac{1}{2}\omega^2M^{-2})\gamma\cdot(p'-k)(\gamma\cdot k)\gamma\cdot(p-k)]$$
$$-M^{-2}[M^2+(p-k)^2]\gamma\cdot(p'-k)(\gamma\cdot k)(\gamma\cdot A)-M^{-2}[M^2+(p'-k)^2](\gamma\cdot A)(\gamma\cdot k)\gamma\cdot(p-k)$$
$$+(1+\mu)[\gamma\cdot A+M^{-2}A\cdot(p'-k)\gamma\cdot(p'-k)](\gamma\cdot k)[\gamma\cdot\omega+M^{-2}\omega\cdot(p-k)\gamma\cdot(p-k)]$$
$$-(1+\mu)[\gamma\cdot\omega+M^{-2}\omega\cdot(p'-k)\gamma\cdot(p'-k)](\gamma\cdot k)[\gamma\cdot A+M^{-2}A\cdot(p-k)\gamma\cdot(p-k)]\}. \quad (6)$$

Asymptotically, $D(k)$ goes as k^{-6} and the numerator function goes as k^4. Thus, S_3 is quadratically divergent at worst. The matrix function $\Sigma(p)$ corresponds to a self-energy diagram for the electron, and diverges quadratically. It would be hoped that the worst divergences would cancel out of the sum of the three diagrams. However, the quadratic divergence coming from the $(1+\mu)$ term in S_3 must remain, since S_1 and S_2 do not depend on μ.

In order to obtain a finite answer for the S-matrix element, a cutoff function must be introduced into the integrals. The simplest covariant cutoff which is sufficiently convergent is, in momentum space, $a^2M^4/(aM^2+k^2)^2$. In the subsequent work this cutoff will be applied to the neutrino propagator. The neutral particle is selected in order to introduce the cutoff in a gauge-invariant manner. Requiring gauge invariance also avoids apparent ambiguities in the magnitude of the cutoff-dependent terms.[4]

An intermediate vector meson must have a mass greater than the K-meson mass.[6] Since $(m_e/m_k)\approx10^{-3}$, it is a very good approximation to take $(m_e/M)^2\ll1$. The further approximation will be made that $(\omega/M)^2\ll1$, with the resulting limit on the range of validity of the results.

The matrix $A_\alpha\Lambda_\alpha$ may be reduced by a consistent procedure to a more transparent, approximate form in terms of the standard integrals

$$I_n(a)=(\pi^2i)^{-1}M^{2n}\int d^4k(k^2+M^2-i\epsilon)^{-(n+2)}\times[a^2M^4(aM^2+k^2-i\epsilon)^{-2}]. \quad (7)$$

The approximation consists in expanding $D(k)$ and the denominator of $\Sigma(p)$ in powers of $(k^2+M^2)^{-1}$ and keeping terms up to first order in (m^2/M^2) and (ω^2/M^2). Products of the two are considered to be of second order. It is necessary to carry first-order terms, because the renormalized S matrix has no terms of order 1.

$$[(p-k)^2+M^2-i\epsilon]^{-1}$$
$$=(k^2+M^2-i\epsilon)^{-1}\times[1+(p^2-2p\cdot k)(k^2+M^2-i\epsilon)^{-1}]^{-1}$$
$$=(k^2+M^2-i\epsilon)^{-1}\{1-(p^2-2p\cdot k)(k^2+M^2-i\epsilon)^{-1}+(p^2-2p\cdot k)^2(k^2+M^2-i\epsilon)^{-2}+\cdots\}. \quad (8)$$

The product of two such series yields the expansion of $D(k)$. One should notice that each successive term leads to a more convergent integral. It is convenient first to obtain a set of standard integrals, done approximately and given for reference in Appendix 1; and then to express $A_\alpha\Lambda_\alpha$ and $\Sigma(p)$ in terms of these integrals. After straightforward application of the free Dirac equation to $A_\alpha\Lambda_\alpha$, [since $S_3\propto\bar{U}(\mathbf{p}')A_\alpha\Lambda_\alpha U(\mathbf{p})$], and the identities $\omega_\alpha A_\alpha(\omega)=0$ and $p\cdot p'=-m^2-\frac{1}{2}\omega^2$, one obtains at some length the following:

$$\Sigma(p)=(g^2/32\pi^2)(1-\gamma_5)(i\gamma\cdot p)[\theta_1+(p^2/M^2)\theta_2]$$
$$A_\alpha(\omega)\Lambda_\alpha(p',p)=-(g^2/32\pi^2)\times\{(\gamma\cdot A)[\theta_1-3(m^2/M^2)\theta_2+(\omega^2/M^2)N_1(\mu)]$$
$$-\gamma_5(\gamma\cdot A)[\theta_1-(m^2/M^2)\theta_2+(\omega^2/M^2)N_1(\mu)]+(m^2/M^2)A_\alpha\sigma_{\alpha\beta}(\omega_\beta/m)N_2(\mu)\}; \quad (9)$$

with

$$\theta_1=-\tfrac{3}{2}(I_{-1}+I_0),\quad \theta_2=\tfrac{1}{2}I_0+I_1+3I_2,$$
$$N_1(\mu)=\tfrac{1}{12}[10I_0-4I_1-6I_2+\mu(3I_{-1}+9I_0-12I_1)],$$

and

$$N_2(\mu)=I_0-I_1+3I_2-\mu(I_0+2I_1).$$

4. RENORMALIZATION OF THE SELF-ENERGY AND VERTEX DIAGRAMS

The matrix elements S_1, S_2, and S_3 may be considered as matrix elements of the self-energy and electrodynamic-vertex parts of the electron to lowest order in the weak interactions. As these matrix elements have been written previously, $\Lambda_\alpha(p',p)$ is the "vertex part" and $\Sigma(p)$ is the "self-energy part", defined for all momenta p and p' by the original integrals in Eq. (5).

[6] In the case $M<m_k$ several of the K-meson decays would occur too rapidly, due to an effective increase in the weak coupling constants.

The vertex part may be separated as follows:

$$\Lambda_\alpha(p',p)=(L_0+L_1\gamma_5)\gamma_\alpha+\Lambda_\alpha{}^f(p',p). \quad (10)$$

The constants L_0 and L_1 may be identified by means of the condition

$$[\Lambda_\alpha{}^f(p',p)]_{i\gamma\cdot p'=i\gamma\cdot p=-m}=0.$$

It should be noted that $\Lambda_\alpha(p,p)$ is not necessarily zero if p is off the mass shell. Applying these conditions to the expression for Λ_α given in Eq. (9) leads to

$$L_0=(g^2/32\pi^2)[-\theta_1+3(m^2/M^2)\theta_2]$$
and
$$L_1=(g^2/32\pi^2)[\theta_1-(m^2/M^2)\theta_2] \quad (11)$$

in terms of θ_1 and θ_2 as previously defined.

In addition, Λ_α as calculated yields the finite part, $\Lambda_\alpha{}^f$, evaluated on the mass shell.

$$[\Lambda_\alpha{}^f(p',p)]_{i\gamma\cdot p'=-m,i\gamma\cdot p=-m}\equiv F_\alpha(p',p)$$
$$=-(g^2/32\pi^2)[(1-\gamma_5)\gamma_\alpha(\omega^2/M^2)N_1(\mu)$$
$$+(m^2/M^2)\sigma_{\alpha\beta}(\omega_\beta/m)N_2(\mu)]. \quad (12)$$

Now the constant terms in the vertex part are usually identified as the coupling constant or charge as modified by the interactions. In the present instance this would involve identifying the term $\gamma_5\gamma_\alpha L_1$ with an induced, quite large pseudocharge or "anapole moment"[4] for the electron. Conversely, to insist that the observed pseudocharge of the electron be zero requires the bare electron to have a pseudocharge. The whole approach will lead to difficulty unless one can argue that the PNC self-energy effects cancel the constant PNC vertex effects. It will be seen that such a cancellation does occur, at least to this order of the theory.

The problem of renormalization of a PNC theory has been treated quite generally by Sekine.[7] In considering a particular model of a PNC theory he concludes that renormalization may be accomplished by introducing a mass shift δm, a wave function renormalization constant Z_2 for the "nucleon" (as well as Z_3 for the "meson" and Z_1 for the coupling constants) and a shift in the representation of the γ matrices. These results are applicable to the case under consideration if the electron is taken as the "nucleon" and the rest of theory is left general in terms of $f(x)$—the right side of the Dirac equation. The equations Sekine derives for δm, Z_2 and $\gamma_\alpha{}^{(+)}$ apply at once, since they follow from the form of the equal-time commutators, CP invariance (and T invariance) and the usual Lehmann-Symanzik-Zimmermann formalism. The actual form of the wave function renormalization procedure is determined by the requirement that the renormalized interacting field and the "in" field have equal matrix elements between the vacuum and a one-particle state. Since the corresponding equation for the unrenormalized interacting field has a term $N'\gamma_5$, the renormalization procedure must include a shift in the representation of the γ matrices (physically, in the definition of the spin states) as well as the usual wave function renormalization constant Z_2.

The improper self-energy part $\Sigma(p)$ may always be written as

$$\Sigma(p)=f_1(p^2)+(i\gamma\cdot p)f_2(p^2)+\gamma_5(i\gamma\cdot p)g(p^2). \quad (13)$$

A term with γ_5 times a function of p^2 is disallowed on the basis of CP invariance. In Sec. 3 these three "form factors" were calculated to order g^2 and have the explicit form

$$f_1{}^{(2)}(p^2)=0,$$
$$f_2{}^{(2)}(p^2)=-g^{(2)}(p^2)=(g^2/32\pi^2)[\theta_1+(p^2/M^2)\theta_2]. \quad (14)$$

To second order in g Sekine's equations for δm, Z_2 and $\gamma_\alpha{}^{(+)}$ yield in terms of $f_1{}^{(2)}$, $f_2{}^{(2)}$ and $g^{(2)}$ the results

$$\delta m=mf_2{}^{(2)}(-m^2)=mL_1,$$
$$Z_2=1+f_2{}^{(2)}(-m^2)-2m^2f_2{}^{(2)\prime}(-m^2)=1-L_0, \quad (15)$$
$$\gamma_\alpha{}^{(+)}\equiv\gamma_\alpha{}^{\rm ren}=\gamma_\alpha[1-g^{(2)}(-m^2)]=\gamma_\alpha(1-L_1\gamma_5).$$

The derivation of Eq. (15) from Sekine's results is given in Appendix 2.

Correct to order g^2 the vertex function $e\Gamma_\alpha(p',p)$ may be written

$$e\Gamma_\alpha(p',p)=e(1+L_0)[\gamma_\alpha(1-L_1\gamma_5)$$
$$+\Lambda_\alpha{}^f(i\gamma\cdot p',i\gamma\cdot p)], \quad (16)$$
$$=e(1+L_0)[\gamma_\alpha{}^{\rm ren}+\Lambda_\alpha{}^f(i\gamma^{\rm ren}\cdot p',i\gamma^{\rm ren}\cdot p)].$$

But wave function renormalization is known to be equivalent to a charge renormalization $e_{\rm ren}=Z_2e$. Again to second order this means $e_{\rm ren}=(1-L_0)e$. It is evident that a further charge renormalization is necessary because of the vertex part just as in ordinary electrodynamics, that is $e_{\rm ren}=(1+L_0)e$. For the physical charge, $e_{\rm phys}$, the total renormalization leads to $e_{\rm phys}=(1+L_0)(1-L_0)e=e$, all to second order.

This cancellation occurs in the usual theory because of Ward's identity. In the present theory the parity-conserving and PNC effects are not separable in their contributions to the renormalization constants. This makes more remarkable the cancellation of the renormalizations of e and the appearance of $\gamma_\alpha{}^{\rm ren}$ in Γ_α. Evidently a more general type of Ward identity (possibly model-dependent) operates in PNC theories of electrodynamics.

The matrix elements S_1 and S_2 can now be considered. For this purpose it is best to rewrite $\Sigma(p)$ in the form

$$\Sigma(p)=\Sigma_0+(m+i\gamma\cdot p)B$$
$$+(m+i\gamma\cdot p)^2\Sigma_f(p)+\gamma_5(i\gamma\cdot p)g(p^2). \quad (17)$$

This amounts to a Taylor expansion of the parity conserving part of Σ about the mass shell. Constants Σ_0 and B are identified by means of the equations

$$\Sigma_0=[\Sigma(p)-\gamma_5(i\gamma\cdot p)g(p^2)]_{i\gamma\cdot p=-m}, \quad (18)$$
$$B=\{d[\Sigma(p)-\gamma_5(i\gamma\cdot p)g(p^2)]/d(i\gamma\cdot p)\}_{i\gamma\cdot p=-m}.$$

[7] K. Sekine, Nuovo Cimento **11**, 87 (1959).

Since $\Sigma(p)-\gamma_5(i\gamma\cdot p)g(p^2)=f_1(p^2)+(i\gamma\cdot p)f_2(p^2)$, the following relations between Σ_0, B, f_1, and f_2 hold:

$$\Sigma_0=f_1(-m^2)-mf_2(-m^2)=-\delta m\,, \qquad (19)$$
$$B=f_2(-m^2)-2m^2f_2'(-m^2)=-L_0\,.$$

In S_2 the self-energy part $\Sigma(p)$ occurs between $S_F(p)$ and $U(p)$ and in S_1, $\Sigma(p')$ between $\bar{U}(p')$ and $S_F(p')$. The constant term Σ_0 in Σ is removed by mass renormalization. The second term is nonsingular but ambiguous and clearly will be incorporated into Z_2 in the usual way. Finally, the PNC term $-S_F(p)\gamma_5(i\gamma\cdot p)g(-m^2)$ becomes simply $\frac{1}{2}\gamma_5 g(-m^2)=-\frac{1}{2}L_1\gamma_5$ since $(m+i\gamma\cdot p)^{-1}\gamma_5=\gamma_5(m-i\gamma\cdot p)^{-1}$. But this term and the corresponding one from S_1 when added to zeroth order Coulomb scattering simply give in place of $\gamma_\alpha A_\alpha(\omega)$ the factor $\gamma_\alpha(1-L_1\gamma_5)A_\alpha(\omega)=\gamma_\alpha{}^{\rm ren}A_\alpha(\omega)$. Thus, the matrix elements S_1 and S_2 are completely accounted for by going to the renormalized mass, charge, and γ representation. The total S-matrix element reduces to the matrix element of only the finite vertex part in terms of the physical mass, charge and γ representation, henceforth symbolized by m, e, γ_α; and the renormalized field by ψ. The absence of a constant γ_5 term in the vertex may be stated as before, that the total pseudocharge induced by the interactions to second order is zero. Although the vertex part does contain a large constant axial-vector term, its magnitude is just such as to be absorbed in the redefinition of the γ matrices.

These considerations lead to the following expression for the renormalized S-matrix element:

$$\begin{aligned}S_{\rm ren}&=-em(2\pi p_0'p_0)^{-1/2}\bar{U}(\mathbf{p}')A_\alpha(\omega)F_\alpha(p',p)U(\mathbf{p})\\&=eg^2m^{\frac{1}{16}}(2\pi^5p_0'p_0)^{-1/2}\bar{U}(\mathbf{p}')A_\alpha(\omega)\\&\quad\times[\tfrac{1}{2}(1-\gamma_5)\gamma_\alpha(\omega^2/M^2)N_1(\mu)\\&\quad+\tfrac{1}{2}(m^2/M^2)\sigma_{\alpha\beta}(\omega_\beta/m)N_2(\mu)]U(\mathbf{p})\,. \qquad (20)\end{aligned}$$

The form for the matrix element agrees with that obtained by Zel'dovich and Perelomov using only invariance properties and the Dirac equation. The two constants may be evaluated as functions of $a^{1/2}$, the cutoff-to-mass ratio, as soon as the integrals $I_n(a)$ are known. They may easily be calculated from the following relationships:

$$J_n(a)=\int_0^\infty dy(1+y)^{-(n+1)}[a/(a+y)]\,, \quad (n\geq 0);$$

and

$$J_0(a)=a(a-1)^{-1}(\ln a)\,.$$

A recursion relation serves to generate the rest of the J_n.

$$J_{n+1}(a)=a(a-1)^{-1}[(n+1)^{-1}-a^{-1}J_n(a)]\,, \quad (n\geq 0)\,.$$

Finally, integrals $I_n(a)$ may be obtained from the $J_n(a)$ according to

$$I_n(a)=a^2[dJ_{n+1}(a)/da]\,, \quad (n\geq -1)\,.$$

5. THE CASE OF A BOUND ELECTRON

In field theory it is possible to define the wave function as the matrix element of the field operator between the appropriate one-particle state and the vacuum. One can then obtain, in the case of an electron, an extended Dirac equation which includes in addition to the external field giving rise to the binding an integral over the T matrix on the appropriate "energy shell" times the wave function. (The external field is assumed to be static.) The new term gives the effect of the other interacting fields present upon a bound electron. Finally, one can define a sort of perturbation theory for the wave function and energy shifts using the $(n-1)$-order correction to ψ with the T matrix to obtain the n-order wave function. It was this sort of procedure which essentially was used in the first calculations of the Lamb shift, according to the equation

$$\Delta E_i{}^{(1)}=\int d^3y\int d^3x\bar{\phi}_i(\mathbf{y})T(\mathbf{y},\mathbf{x};\,E_i{}^{(0)},E_i{}^{(0)})\phi_i(\mathbf{x})\,. \quad (21)$$

Here the functions ϕ_i are the solutions of the Dirac equation in the external field.

A similar equation may be obtained for mixing amplitudes due to the interacting fields, just as in ordinary perturbation theory.

$$\eta_{ij}=(\Delta E_{ij}{}^{(0)})^{-1}\int d^3y\int d^3x\bar{\phi}_j(\mathbf{y})\times T(\mathbf{y},\mathbf{x};\,E_j{}^{(0)},E_i{}^{(0)})\phi_i(\mathbf{x})\,, \quad (22)$$

where

$$\psi_i{}^{(1)}(\mathbf{x})=\phi_i(\mathbf{x})+\sum_{j\neq i}\eta_{ij}\phi_j(\mathbf{x})\,.$$

The calculation in Secs. 1–4 yields the T matrix on the free-particle mass shell from the S matrix according to $S=1-2\pi iT\delta(p_0'-p_0)$. Thus, in the present case T is given by

$$\begin{aligned}&T(\mathbf{p}',\mathbf{p};\,p_0',p_0)\delta(\omega_0)\\&\quad=-iem(8\pi^3p_0'p_0)^{-1/2}\bar{U}(\mathbf{p}')F_\alpha(p',p)A_\alpha(\omega)U(\mathbf{p})\,. \qquad (23)\end{aligned}$$

Until now the only approximations have been second-order field-theoretic perturbation theory for T and $(m^2/M^2)\ll 1$, $(\omega^2/M^2)\ll 1$. At this point two additional approximations will be introduced—that of first-order perturbation theory for the wave function and that of replacing T on the bound-electron "energy shells" by T on the free-particle mass shells. The first of the two is expected to be valid, especially in view of the small η which will be obtained. As for the second, the energy of the bound states is $E_i{}^{(0)}=m(1-e^4/n_i{}^2)$ and the momenta characteristic of the Fourier components of the bound states are about $1/a_0=me^2$, hence $(\mathbf{p}^2+m^2)^{1/2}\approx m(1+\frac{1}{2}e^4)$. (That also means that the values of ω^2 which are important are less than m^2.) Therefore, the error should not be too large unless T should happen to be a rapidly varying function of energy near m.

The PNC term in T may be obtained now from Eq. (20).

$$T_{\mathrm{PNC}}\delta(\omega_0)=ieg^2m^{\frac{1}{2}}(8\pi^7p_0'p_0)^{-1/2}\times\bar{U}(\mathbf{p}')(A\cdot\gamma)\gamma_5(\omega^2/M^2)U(\mathbf{p})N_1(\mu). \quad (24)$$

In configuration space $\Box A_\alpha(x)=-j_\alpha(x)$ corresponds to $A_\alpha(\omega)\omega^2=j_\alpha(\omega)$ in momentum space. Since the external current responsible for atomic binding is approximately one point charge at the nucleus, the coefficients η_{ij} will depend only on the wave functions and their derivatives at the nucleus (or nuclei in the case of a molecule).

Returning to Eq. (22), one may now transform from the variables $(\mathbf{y},\mathbf{x})$ to $(\mathbf{p}',\mathbf{p})$ according to

$$T(\mathbf{y},\mathbf{x};E_2,E_1)=(2\pi)^{-3}\int d^3p'\int d^3p\times T(\mathbf{p}',\mathbf{p};E_2,E_1)e^{i\mathbf{p}'\cdot\mathbf{y}}e^{-i\mathbf{p}\cdot\mathbf{x}}. \quad (25)$$

This relation enables the expression for η_{ij} to be rewritten

$$\eta_{ij}\approx(\Delta E_{ij}^{(0)})^{-1}\int d^3p'\int d^3p\bar{\phi}_j(\mathbf{p}')\times T(\mathbf{p}',\mathbf{p};p_0',p_0)\phi_i(\mathbf{p}). \quad (26)$$

Since the momentum functions $U_s(\mathbf{p})$ and $V_s(\mathbf{p})$ for $s=1$, 2 form a complete set with respect to spin functions, the PNC amplitudes may be written in the special case at hand

$$\eta_{ij}^{\mathrm{PNC}}\approx C(\Delta E_{ij}^{(0)})^{-1}\int d^3p'\int d^3p\times\bar{\phi}_j(\mathbf{p}')\gamma_5\omega^2A(\omega)\cdot\gamma\phi_i(\mathbf{p}), \quad (27)$$

where

$$C=-\tfrac{1}{8}i(2\pi)^{-7/2}e(g^2/M^2)N_1(\mu).$$

Further, since

$$\omega^2\gamma\cdot A(\omega)=(2\pi)^{-3/2}\int d^3xe^{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\gamma\cdot j(\mathbf{x}),$$

the following equation also holds:

$$\eta_{ij}^{\mathrm{PNC}}\approx C(\Delta E_{ij}^{(0)})^{-1}\int d^3x\bar{\phi}_j(\mathbf{x})\times[(2\pi)^{3/2}\gamma_5\gamma\cdot j(\mathbf{x})]\phi_i(\mathbf{x}). \quad (28)$$

In the case of a hydrogen-like atom the external current is taken to be a δ function at the nucleus; $\gamma\cdot j(\mathbf{x})=i\gamma_4Ze\delta(\mathbf{x})$; and the expression for the mixing parameters due to PNC effects is

$$\eta_{ij}^{\mathrm{PNC}}\approx(\Delta E_{ij}^{(0)})^{-1}(Ze^2/32\pi^2)(g^2/M^2)N_1(\mu)\times\bar{\phi}_j(\mathbf{0})\gamma_5\gamma_4\phi_i(\mathbf{0}). \quad (29)$$

The factor $\bar{\phi}(\mathbf{0})\gamma_5\gamma_4\phi_i(\mathbf{0})$ will vanish in the nonrelativistic limit because γ_5 is odd and γ_4 even, i.e., the matrix couples large and small components. In order to get the dominant contribution to this expression, one can proceed as follows, (assuming that $[\nabla\phi_i]_{\mathbf{x}=0}=0$ but $\phi_i(0)\neq0$ e.g., when ϕ_i is an S state):

$$\bar{\phi}_j(\mathbf{x})e^{-iEt}(m+eV(|\mathbf{x}|)\gamma_4)-\partial_\mu\bar{\phi}_j(\mathbf{x})e^{-iEt}\gamma_\mu=0, \quad (30)$$

where $V(|\mathbf{x}|)$ is the electrostatic potential responsible for the binding. When the Dirac equation is solved for $\bar{\phi}_j(\mathbf{x})$ and $\mathbf{x}$ is set equal to 0, remembering that $V(|\mathbf{x}|)\bar{\phi}_j(\mathbf{x})$ remains bounded when $\nabla\bar{\phi}_j$ is nonzero at $\mathbf{x}=0$, a new expression for $\bar{\phi}_j(\mathbf{0})$ is obtained.

$$\bar{\phi}_j(\mathbf{0})=m^{-1}[\nabla\bar{\phi}_j(\mathbf{x})]_{\mathbf{x}=0}\cdot\gamma-(E_j/m)\bar{\phi}_j(\mathbf{0})\gamma_4-(\mathrm{e/m})[\mathrm{V}(|\mathbf{x}|)\bar{\phi}_j(\mathbf{x})]_{\mathbf{x}=0}\gamma_4. \quad (31)$$

Now $\gamma_4\gamma_5\gamma_4$ is an odd operator and $\gamma\gamma_5\gamma_4$ is even. Thus, the first term in Eq. (31) couples large to large components and is dominant.

$$\bar{\phi}_j(\mathbf{0})\gamma_5\gamma_4\phi_i(\mathbf{0})\approx m^{-1}[\nabla\bar{\phi}_j(\mathbf{x})]_{\mathbf{x}=0}\cdot[\gamma\gamma_5\gamma_4\phi_i(\mathbf{0})]\approx im^{-1}[\nabla\phi_j^{\dagger}(\mathbf{x})]_{\mathbf{x}=0}\cdot[\sigma\phi_i(\mathbf{0})], \quad (32)$$

where in the last expression $\phi_j^{\dagger}$ and ϕ_i are 2-component nonrelativistic hydrogen wave functions.

This discussion may now be specialized to two specific states in atomic hydrogen in order to treat a definite and interesting case. The 2^2S and 2^2P states with wave functions $\phi_s(\mathbf{x})$ and $\phi_p(\mathbf{x})$ are split in energy by the Lamb shift due to the usual radiative corrections in electrodynamics; quantitatively $\Delta E_L=E_s^{(0)}-E_p^{(0)}=4.374\times10^{-6}$ eV. (It is expected that the further corrections to this splitting due to the weak interactions will be very small.) Since the energy denominator for any state other than 2^2P will be larger by about 5×10^5, it will suffice to consider only this state and the corresponding η_{ij}.

$$\psi_s^{(1)}(\mathbf{x})\approx\phi_s(x)+\eta\phi_p(\mathbf{x}) \quad (33)$$

with

$$\eta=(\Delta E_L)^{-1}(ie^2/32\pi^2m)(g^2/M^2)[\nabla\phi_p^{\dagger}(\mathbf{x})]_{\mathbf{x}=0}\cdot[\sigma\phi_s(\mathbf{0})],$$

where $\phi_p^{\dagger}(\mathbf{x})$ and $\phi_s(\mathbf{x})$ are just the nonrelativistic wave functions.

A set of good quantum numbers for these two states are n, j, m_j and parity. "S" is understood to refer to the even parity states with $n=2$, $j=\frac{1}{2}$, $m_j=\pm\frac{1}{2}$; and "P" to the odd parity states with the other quantum numbers the same. The operator T, of course, will connect only states of equal j and m_j, because the atom is considered as an isolated system. Thus, the wave functions will be

$$\phi_s^{(\pm1/2)}(\mathbf{x})=(8a_0^5)^{-1/2}(|\mathbf{x}|-2a_0)e^{-|\mathbf{x}|/2a_0}Y_0^{(0)}(\hat{x})\chi_{\pm1}$$
$$\phi_p^{(\pm1/2)}(\mathbf{x})=(72a_0^5)^{-1/2}|\mathbf{x}|e^{-|\mathbf{x}|/2a_0}\times[Y_1^{(0)}(\hat{x})\chi_{\pm1}-\sqrt{2}Y_1^{(\pm1)}(\hat{x})\chi_{\mp1}]. \quad (34)$$

From these wave functions and Eq. (33) it is quite straightforward to obtain an explicit value for η.

$$|\eta|=(\tfrac{3}{2})^{1/2}(\alpha^5m^3/32\pi^2)GN_1(\mu)(\Delta E_L)^{-1},\approx2.808\times10^{-14}N_1(\mu). \quad (35)$$

Figure 2 shows a log by log graph of the absolute

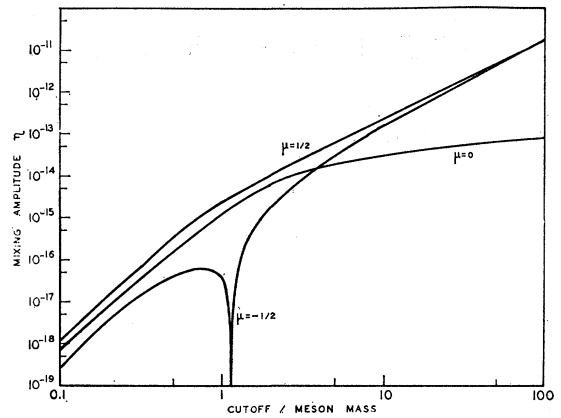

FIG. 2. Absolute value of the amplitude η of 2^2P state present in the predominantly 2^2S state of hydrogen versus the ratio $a^{1/2}$ of the cutoff to meson mass. The three curves correspond to different values for the anomalous gyromagnetic ratio μ of the meson. For $\mu=-\frac{1}{2}$ the amplitude η actually vanishes slightly above $a^{1/2}=1$.

value of η versus the cutoff-to-mass ratio $a^{1/2}$ for various values of the anomalous gyromagnetic ratio μ.

For values of $a^{1/2}$ between 10 and 100, and for $\mu\approx 0$, $|\eta|$ is much smaller than the 10^{-9} predicted by Zel'dovich and Perelomov. This probably means that the vector-meson theory contains cancellations for the scattering process considered which are not expected from general considerations.

6. DISCUSSION

A more exact calculation of the parity-nonconserving effects would involve the use of a bound-interaction formalism including the vector-meson field as well as the electron field. In the usual calculations of the Lamb shift such a formalism for the electron field is the starting point. Approximations are then introduced, but they are more explicitly justified than those employed in Sec. 5. Alternatively, one could recalculate the vertex diagram off the mass shell, and obtain the correct T-matrix elements; but this would still neglect the effect of the external field on the meson in the intermediate state.

It is clear that the $2S$-$2P$ parity impurities in hydrogen are much too small to be seen experimentally. The electric field which would mix these states with amplitude 10^{-9} due to the linear Stark effect is only 4.50×10^{-5} V/m. To reduce stray fields beyond this amount would be difficult. Also, since the hydrogen molecule has quite different electronic states, atomic hydrogen would have to be employed.

Although such an experiment would be impractical, a more feasible attempt to measure parity impurities in bound electron states has been made by Bradley and Wall.[8] From the absence of circular dichroism in an absorption line of the O_2 molecule (the transition was carefully selected to give an effective amplification to any such effect), they set an upper limit on the parity impurity amplitude of the ground state. They assert that $|\eta|\leq 3\times10^{-8}$, where the ground state of O_2 is $\psi_0=|1-\eta^2|^{1/2}\psi^{(+)}+\eta\psi^{(-)}$; $\psi^{(+)}$ and $\psi^{(-)}$ are the even and odd parity parts of ψ_0, respectively.

The known low-lying excited states of O_2 with odd parity are $^3\Delta_u$ with $\Delta E=4.30$ eV, $^1\Sigma_u^-$ with $\Delta E=4.55$ eV, $^3\Sigma_u^+$ with $\Delta E=4.67$ eV and $^3\Sigma_u^-$ with $\Delta E=6.44$ eV; all energy differences are relative to the ground state. These states all have configuration $(1\sigma_g)^2(1\sigma_u)^2(2\sigma_g)^2\times(2\sigma_u)^2(3\sigma_g)^2(1\pi_u)^3(1\pi_g)^3$, whereas the ground-state configuration is $\cdots(1\pi_u)^4(1\pi_g)^2$. The Balmer states all have energies near these, but the electron amplitude at the origin becomes negligible as the orbit "radii" become large. States with dissociation products other than $^3P+^3P$ are higher in energy—20 eV or more from the ground state.

Again resorting to perturbation theory, one can write the ground-state wave function as

$$\psi_0=\psi(^3\Sigma_g^-)+\beta_1\psi(^1\Delta_g)+\eta_1\psi(^3\Delta_u)+\cdots. \quad (36)$$

The properly normalized wave function Ψ_0 is $\Psi_0=(\sqrt{\langle\psi_0|\psi_0\rangle})^{-1}\psi_0$, but $\langle\psi_0|\psi_0\rangle\approx 1$, since the β_i and η_i are very small. The effective impurity amplitude η will be overestimated by $\sum_i|\eta_i|$. Taking for the operator of the interaction a sum of single-particle operators

$$O_i=(Ze^2/32\pi^2m)(g^2/M^2)\boldsymbol{\sigma}_i\cdot[\nabla_i\delta(\mathbf{x}_i)+\nabla_i\delta(\mathbf{x}_i-\mathbf{a})],$$

(where $\mathbf{a}$ is the vector from one nucleus to the other) and a properly antisymmetrized LCAO wave function, one might hope to obtain an estimate of, say, η_1 by setting

$$\eta_1=(\Delta E_1)^{-1}\int d^3x\phi_{\rm exc}^{\dagger}(\mathbf{x})O(\mathbf{x})\phi_{\rm gnd}(\mathbf{x}). \quad (37)$$

Here $\phi_{\rm exc}$ and $\phi_{\rm gnd}$ are the space-spin wave functions of the excited electron in the $^3\Delta_u$ and $^3\Sigma_g^-$ states, respectively. Unfortunately, for all of the listed states $\phi_{\rm exc}$ and $\phi_{\rm gnd}$ vanish at $\mathbf{x}=0$ and $\mathbf{x}=\mathbf{a}$, because they are formed by excitation of a π electron. This fact means that the matrix elements are zero with simple LCAO wave functions, and that to estimate them is a more difficult task. It may even be that the lowest lying Balmer state of the correct type will have the largest contribution.

Assuming that the amplitudes at the nuclei of the excited electron in excited states and the ground state are about the same as before, one would roughly expect for $\eta(O_2)$ relative to that obtained for hydrogen, $\eta(H)$, the value

$$\eta(O_2)\approx[(\Delta E_L/\Delta E)(2Z)N]\eta(H). \quad (38)$$

The energy difference ΔE is some average for the N states which contribute more or less equally. The factor $2Z$ comes from the strength of the external field. If $\Delta E=4$ eV, $N=10$, $\mu=0$ and $a^{1/2}=100$, then $\eta(O_2)\approx10^{-18}$. Because the matrix elements of T vanish approximately

[8] L. Bradley and N. Wall, Nuovo Cimento **25**, 48 (1962).

for the low-lying states, the actual value for η, if calculated carefully, would be considerably smaller.

Thus, the vector-meson theory suggests a value for the parity impurity in the ground state of O_2 which is far below the experimental upper limit of 3×10^{-8}. Further, a direct coupling theory like the Fermi theory is expected to give a result smaller by about 10^{-15}. Thus, even a considerable refinement of the experiment on O_2 would probably not reveal the PNC effects present.

But, now that the factors which affect the impurity amplitude have been elucidated assuming a vector meson, it should be possible to devise an experimental situation with a much larger expected impurity.

In particular it may be said that the parity impurity of an atomic or molecular state is proportional to Z, inversely proportional to the energy *gap* between the state and states of opposite parity (but *not* necessarily the energy difference of the *transition* used in the experiment) and proportional to the amplitude at the nucleus or nuclei of the wave function of the electron involved. In the case of hydrogen-like atoms (or L shells of heavy nuclei), for example, η goes as Z since ΔE_L goes as Z^4.

A thorough search might reveal an atomic state in which η is considerably larger than in H or O_2, perhaps even in the range 10^{-9} to 10^{-11}. A circular dichroism experiment might then be feasible with greater chance of observing an effect.

The renormalized S-matrix element given in Eq. (20) has more general validity than has been indicated. It has been derived assuming that $A_\alpha(\omega)$ represents the Fourier transform of an external electromagnetic field expressed in Lorentz gauge. But by a simple extension it may be applied to electron scattering by any charged particle to the accuracy of one-quantum exchange. To do this one need only take A_α proportional to the Fourier transform of the electric current corresponding to the other charged particle.

Thus, if the other particle is a positron and the initial and final momenta are q and q' respectively, then Eq. (20) applies, taking $A_\alpha(\omega)=\delta(p+q-p'-q')ie(2\pi)^{-3/2}\times m(q_0'q_0)^{-1/2}(\omega^2-i\epsilon)^{-1}\bar{V}(\mathbf{q})\gamma_\alpha V(\mathbf{q}')$. This expression is obtained by applying the Feynman rules to an expanded diagram with the external field replaced by an internal photon line and the appropriate positron vertex.

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APPENDIX 1

$$J(v)=\int d^4k[M^2+(p-k)^2-i\epsilon]^{-1}(k-i\epsilon)^{-1}(v\cdot k),$$

$$J(v,w)=\int d^4k[M^2+(p-k)^2-i\epsilon]^{-1}(k^2-i\epsilon)^{-1}(v\cdot k)(w\cdot k),$$

$$J'(v)=\int d^4k[M^2+(p-k)^2-i\epsilon]^{-1}(v\cdot k),$$

$$A(v)=\int d^4kD(k)(v\cdot k),$$

$$A(v,w)=\int d^4kD(k)(v\cdot k)(w\cdot k),$$

$$B(v)=\int d^4kD(k)k^2(v\cdot k),$$

$$B(v,w)=\int d^4kD(k)(v\cdot k)(w\cdot k)k^2.$$

These equations define the integrals used in the calculation. To first order in m^2/M^2 or ω^2/M^2 they have the following expressions in terms of the standard integrals I_n, $(n\geq-1)$:

$$J(v)=\tfrac{1}{2}\pi^2i(v\cdot p)[I_0-2(p^2/M^2)I_2],$$
$$J(\gamma,\gamma)=\pi^2iM^2[I_{-1}-(p^2/M^2)I_1],$$
$$J'(v)=\tfrac{1}{2}\pi^2i(v\cdot p)M^2[(I_{-1}-I_0)-2(p^2/M^2)(I_1-I_2)].$$

The second integral is specialized to the one actually encountered in $\Sigma(p)$. All of the three are expressed in terms of p^2—off the mass shell. Since they are used in expanding $\Sigma(p)$, the p dependence must be explicit.

$$A(v)=\tfrac{1}{2}\pi^2iv\cdot(p+p')M^{-2}[I_1-(m^2/M^2)(I_2-4I_3)-\tfrac{2}{3}(\omega^2/M^2)(I_2-I_3)],$$
$$A(v,w)=\tfrac{1}{4}\pi^2i(v\cdot w)[I_0+2(m^2/M^2)I_2-\tfrac{1}{3}(\omega^2/M^2)(I_1-I_2)]+\tfrac{1}{6}\pi^2iM^{-2}[v\cdot(p+p')w\cdot(p+p')+(v\cdot p)(w\cdot p)+(v\cdot p')(w\cdot p')](I_1-I_2),$$
$$B(v)=\tfrac{1}{2}\pi^2iv\cdot(p+p')[(I_0-I_1)-(m^2/M^2)(I_1-5I_2+4I_3)-\tfrac{2}{3}(\omega^2/M^2)(I_1-2I_2+I_3)],$$
$$B(v,w)=\tfrac{1}{4}\pi^2i(v\cdot w)M^2[(I_{-1}-I_0)+2(m^2/M^2)(I_1-I_2)-\tfrac{1}{3}(\omega^2/M^2)(I_0-2I_1+I_2)]+\tfrac{1}{6}\pi^2i[v\cdot(p+p')w\cdot(p+p')+(v\cdot p)(w\cdot p)+(v\cdot p')(w\cdot p')](I_0-2I_1+I_2).$$

APPENDIX 2

A few steps are necessary in going from the equations obtained by Sekine to the second-order expressions given in Eq. (15). First, it is convenient to reproduce here his relevant definitions and results. Three functions $\Sigma_i(p^2)$ with $i=1, 2, 3$ are defined by

$$\langle0|\{f(x),\bar{f}(y)\}|0\rangle=-(2\pi)^{-3}\int d^4pe^{ip\cdot(x-y)}\epsilon(p)\times[\Sigma_1+(m+i\gamma\cdot p)\Sigma_2+\gamma_5(i\gamma\cdot p)\Sigma_3], \quad \text{(A2.1)}$$

where $(\gamma_\mu\partial_\mu+m)\psi(x)=f(x)$ and m is the physical mass.

Constants N and N' are defined from

$$\langle 0|\psi(x)|p\rangle=(1+N+N'\gamma_5)\langle 0|\psi^{(0)}(x)|p\rangle, \quad \text{(A2.2)}$$

where $\psi(x)$ is the interacting Heisenberg field and $\psi^{(0)}(x)$ is the "in" field.

Sekine proves this equation relating the matrix elements of the respective operators between a physical one-particle state and the physical vacuum. In terms of these definitions and a calculation of the equal-time commutators of the fields ψ, he concludes that

$$\begin{aligned} N&=1-\tfrac{1}{2}[\bar{\Sigma}_2(-m^2)+2m\bar{\Sigma}_1'(-m^2)],\\ N'&=\tfrac{1}{2}\bar{\Sigma}_3(-m^2),\\ \delta m&=\bar{\Sigma}_1(-m^2), \end{aligned} \quad \text{(A2.3)}$$

where

$$\bar{\Sigma}_i(p^2)=\int_0^\infty da(p^2+a)^{-1}\Sigma_i(-a) \quad \text{for} \quad p^2\approx -m^2;$$

[remembering, here, that $\Sigma_i(p^2)=0$ for $-p^2$ less than the mass of the first intermediate state].

Finally, the renormalization constant Z_2 and the shift in γ representation are defined in terms of N and N' by

$$Z_2=N^2-N'^2$$

$$\gamma_\mu^{(+)}=\mathfrak{M}\gamma_\mu\mathfrak{M}^{-1} \text{ where } \mathfrak{M}=(Z_2)^{-1/2}(N+N'\gamma_5). \quad \text{(A2.4)}$$

It is easily verified that $\{\gamma_\mu^{(+)},\gamma_\nu^{(+)}\}=Z_2^{-1}(N^2-N'^2)\times\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu}$. (If the γ_μ are taken as Hermitian, then the $\gamma_\mu^{(+)}$ are no longer Hermitian due to the use of a non-unitary similarity transformation.)

The functions Σ_i are all of order g^2 and higher. Thus, to second order, Eq. (A2.3) becomes

$$\begin{aligned} \delta m&=\bar{\Sigma}_1^{(2)}(-m^2),\\ Z_2&=1-\bar{\Sigma}_2^{(2)}(-m^2)-2m\bar{\Sigma}_1^{(2)\prime}(-m^2),\\ \gamma_\mu^{(+)}&=(1+2N'\gamma_5)\gamma_\mu=[1+\bar{\Sigma}_3^{(2)}(-m^2)\gamma_5]\gamma_\mu. \end{aligned} \quad \text{(A2.5)}$$

Now the improper self-energy part $\Sigma(p)$ may be written in the absence of electron self-interactions as

$$\Sigma(p)=i\int d^4x e^{-ip\cdot x}\langle 0|\mathbf{T}\{f(\tfrac{1}{2}x)\bar{f}(-\tfrac{1}{2}x)\}|0\rangle. \quad \text{(A2.6)}$$

In terms of the Σ_i, therefore, $\Sigma(p)$ has the form

$$\Sigma(p)=-\int_0^\infty da(p^2+a-i\epsilon)^{-1}[\Sigma_1(-a)+(m+i\gamma\cdot p)\Sigma_2(-a)+\gamma_5(i\gamma\cdot p)\Sigma_3(-a)]. \quad \text{(A2.7)}$$

The following identifications become at once possible:

$$\begin{aligned} f_1(p^2)&=-\int_0^\infty da(p^2+a-i\epsilon)^{-1}[\Sigma_1(-a)+m\Sigma_2(-a)],\\ f_2(p^2)&=-\int_0^\infty da(p^2+a-i\epsilon)^{-1}\Sigma_2(-a),\\ g(p^2)&=-\int_0^\infty da(p^2+a-i\epsilon)^{-1}\Sigma_3(-a). \end{aligned} \quad \text{(A2.8)}$$

Also, because $\Sigma_i(p^2)$ vanish for p^2 in some neighborhood of $-m^2$, one obtains the following relationships for $p^2\approx -m^2$:

$$\begin{aligned} f_1^{(2)}(p^2)&=-[\bar{\Sigma}_1^{(2)}(p^2)+m\bar{\Sigma}_2^{(2)}(p^2)],\\ f_2^{(2)}(p^2)&=-\bar{\Sigma}_2^{(2)}(p^2),\\ g^{(2)}(p^2)&=-\bar{\Sigma}_3^{(2)}(p^2). \end{aligned} \quad \text{(A2.9)}$$

But f_1 vanishes to second order, and this implies

$$\bar{\Sigma}_1^{(2)\prime}(-m^2)=-m\bar{\Sigma}_2^{(2)\prime}(-m^2). \quad \text{(A2.10)}$$

From these equations and Eq. (A2.5) the results stated in Eq. (15) follow trivially.